MATH 2040 A Lecture 2 (Sep 12, 2016)

Revision II (Textbook Ch. 1–4)

Recall: $\beta \subseteq V$ basis

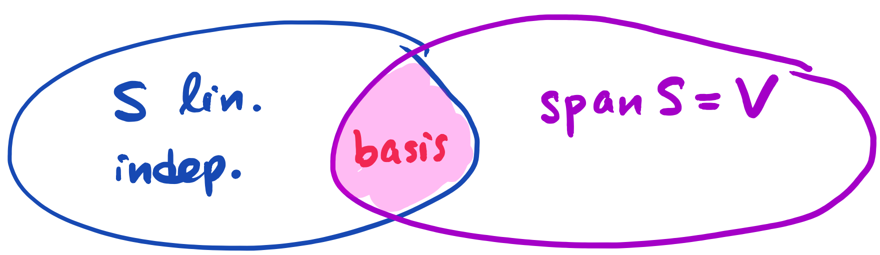
$\boxed{\dim V = |\beta|}$

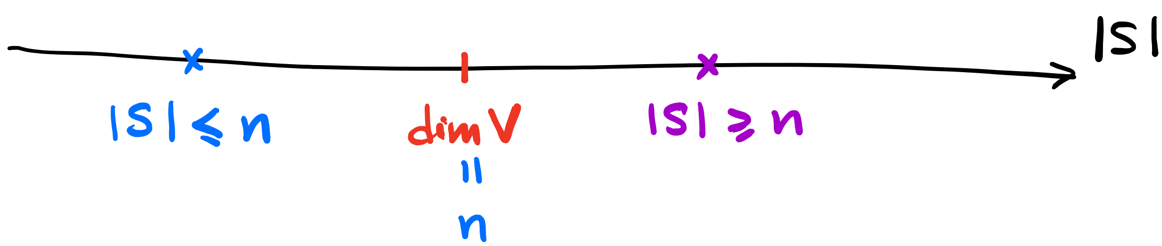
# of vectors in β

$\iff$ ① $\operatorname{span}(\beta) = V$

② β lin. indep.

$S \subseteq V$ subset of vectors





Thm: V vector space / $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$)

$\dim V = n < +\infty$.

(a) $S \subseteq V$ lin. indep. $\Rightarrow$ $|S| \leq n$

(b) $\operatorname{span} S = V$ $\Rightarrow$ $|S| \geq n$

Thm: (a) Any lin. indep $S \subseteq V$ can be extended to a basis of V.

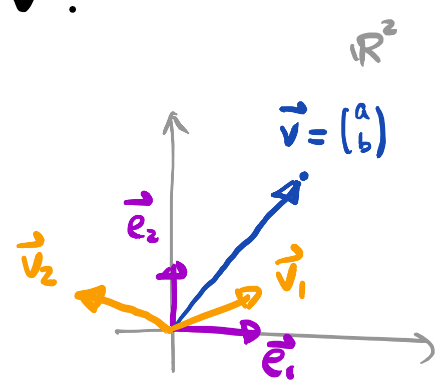
$$\underset{\text{lin. indep.}}{S = \{\vec{v}_1, \dots, \vec{v}_k\}} \longrightarrow \underset{\text{basis}}{\beta = \{\overbrace{\vec{v}_1, \dots, \vec{v}_k}^{S}, \vec{v}_{k+1}, \dots, \vec{v}_n\}}$$

$(k \leq n)$

(b) $W \subseteq V$ subspace $\Rightarrow \dim W \leq \dim V$

"=" holds $\Leftrightarrow$ $W = V$.

Example: $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} = a \cdot \vec{e}_1 + b \cdot \vec{e}_2$



$$\vec{v} \overset{\cong}{\longleftrightarrow} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\vec{v} = c\,\vec{v}_1 + d\,\vec{v}_2$$

$$\vec{v} \overset{\cong}{\longleftrightarrow} \begin{pmatrix} c \\ d \end{pmatrix}$$

Remark: $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ basis for V $(\dim V = n)$

For each $\vec{v} \in V$, $\exists$ unique $a_1, \dots, a_n \in \mathbb{F}$ s.t.

$$\boxed{\vec{v} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n}$$

$$V \overset{\cong}{\longleftrightarrow} \mathbb{F}^n$$

$$\vec{v} \overset{\cong_\beta}{\longleftrightarrow} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

§ System of Linear Equations

m equations, n unknowns $(x_1, \ldots, x_n)$

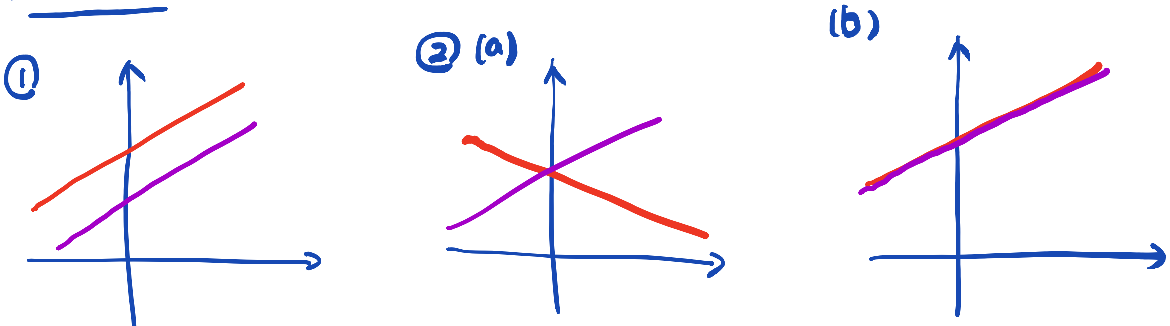
$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

3 possibilities
- ① NO SOLUTION
- ② $\exists$ SOLUTION
  - (a) ONLY 1 SOLUTION "best"
  - (b) $\exists$ ∞'ly many solutions

Geometrically, $(*) \iff$ intersection of lines/planes (hyperplanes in $\mathbb{R}^n$)

<u>$n = m = 2$</u> :

① ② (a) (b)



NO SOLUTION

UNIQUE SOL$^{\underline{n}}$ "generic"

∞'LY MANY SOL$^{\underline{n}}$

Written in terms of matrix / vectors

$$(*) \iff \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}}$$

A: $m \times n$ matrix (rows: m, col$^{\underline{n}}$: n)

$\vec{x} \in \mathbb{R}^n$, $n \times 1$

$\vec{b} \in \mathbb{R}^m$, $m \times 1$

$$(*) \iff \boxed{A\vec{x} = \vec{b}}$$

Idea: " A^{-1} exists " $\Rightarrow$ $\vec{x} = A^{-1}\vec{b}$.

$(M_{m\times n}(\mathbb{F}), +, \cdot)$ is a vector space $/\mathbb{F}$.
(dim = mn)

Matrix multiplication:

$$\underset{m\times n}{A}\ \underset{n\times k}{B} = \underset{m\times k}{C}$$

- $A(B+C) = AB + AC$ etc.

Caution: $AB \neq BA$ (non-commutativity)

but $(AB)C = A(BC)$ o.k.

Def$^{\underline{n}}$: $A \in M_{n\times n}(\mathbb{F})$ is invertible

if $\exists\ A^{-1} \in M_{n\times n}(\mathbb{F})$ s.t.

"the" inverse of A

$$A^{-1}A = I \;(= AA^{-1})$$

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

"identity matrix"

$IA = A$ $\forall A$.

Question: When is A invertible?

(Quick) **Ans**: use "determinants".

Thm: $A \in M_{n\times n}(\mathbb{F})$ invertible $\iff$ "$\det A \neq 0$."

§ Determinants

$A \in M_{n\times n}(\mathbb{F})$, define "recursively" (on n)

$$(\#) \quad \boxed{\det A := \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})}$$

"expansion along i^{th} row"

Remark: freedom to pick i^{th} row.

n=1: $A = (a)$ $\qquad \det A = a$

$$\begin{cases} ax = b \\ \Rightarrow x = a^{-1} b \end{cases}$$

$a \neq 0!$

n=2: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = ad - bc$

$$\parallel$$

(#): $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\, d \;-\; b\, c$

n=3:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = -\, a_{21} \cdot \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix}$$

$$+\, a_{22} \cdot \det \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix}$$

$$-\, a_{23} \cdot \det \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & \cdots & \\ \cdots & & & \end{pmatrix}$$

Theorems: (A) ☆ $\boxed{\det(AB) = \det(A)\det(B)}$ ☆

Ex / Caution: $\det(A+B) \neq \det(A) + \det(B)$

$\det(cA) \neq c \det(A)$, $c \in \mathbb{F}$

But: $\det(cA) = c^n \det(A)$

$A \in M_{n \times n}(\mathbb{F})$

(B) $\det(A^T) = \det(A)$

$\uparrow$ transpose of A

E.g. $\det\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

(A) $\Rightarrow$ Thm: $\det A \neq 0 \Leftarrow A$ invertible

"Quick Proof": A invertible

By def^{n}: $\exists A^{-1}$ s.t. $A^{-1}A = I$

$$\det(A^{-1}A) = \det I = 1$$

$$\det(A^{-1})\det(A) \Rightarrow \det(A) \neq 0.$$

$\square$

Problem: If $A \in M_{n\times n}(\mathbb{F})$ s.t. $\det A \neq 0$,

then how to find A^{-1}?

$$(A \mid I) \xrightarrow[\text{operations"}]{\text{"row}} \underbrace{(I \mid A^{-1})}_{\text{RREF of } (A \mid I)}.$$

§ Row operations & RREF.

- Elementary row op. (col.)
  (I) $(i) \longleftrightarrow (j)$
  (II) $\alpha (i)$, $\alpha \in \mathbb{F}$
  (III) $\alpha (i) + (j)$

- Elementary matrices: applying (I), (II), (III) to I.

  E.g: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \xleftarrow{r(1)+r(2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

"Elem. row (col.) operations" $\Longleftrightarrow$ "Multiply by elem. matrices from the left (right)"

By elem. row op., any matrix has a unique RREF (reduced row echelon form)

$$R = \begin{pmatrix} 1 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * & 0 & * \\ \vdots & \vdots & & & 1 & * & 0 & * \\ \vdots & \vdots & & 0 & & & 1 & * \\ 0 & 0 & & & & 0 & & 0 \end{pmatrix}$$

"pivots"

$\Rightarrow$ rank(A) = # of "pivots"

$= \dim(\underbrace{C(A)}_{\text{column}})$

column space of A $:= \operatorname{span}(\vec{a}_1, \dots, \vec{a}_n)$ $\qquad A = \begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix}$

<u>FACTs</u>: • Every invertible $A \in M_{n\times n}(\mathbb{F})$ can be written as

$$A = \underbrace{E_1 E_2 \cdots E_k}_{\substack{\text{product of} \\ \text{elementary} \\ \text{matrices}}}$$

• Let P, Q be invertible. Then

$$\operatorname{rank}(PAQ) = \operatorname{rank}(A)$$